FINITE DIFFERENCE SCHEMES FOR THE POISSON EQUATION

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Abstract

In this paper, we review two popular finite difference schemes for solving the Poisson equation in 2-D domains. First, we introduce the standard five-point discretization scheme. An error analysis is also given to show that the accuracy of the five-point scheme is $O(h^2)$, where h is the grid size of the spatial discretization. We then introduce a nine-point finite difference scheme which is $O(h^4)$ accurate. Numerical experiments are carried out to demonstrate the theoretical analysis.

Keywords: Poisson equation, finite difference scheme, error estimate, 2-norm, eigenvalue

AMS subject classifications: 65N06; 65N12; 65N15

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FINITE DIFFERENCE SCHEMES FOR THE POISSON EQUATION

波松方程的有限差分法

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摘要

本論文中介紹兩種常用來求解二維波松方程的有限差分法。首先我們介紹標準的五點有限差分格式，同時提供其誤差分析，我們證明此五點格式的精度是二階收斂。緊接著我們介紹另一種四階收斂的九點有限差分格式。數值實驗驗證了此兩種有限差分格式的理論分析。

關鍵詞：波松方程、有限差分法、誤差分析、2-模、特徵值

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1. Introduction

In this paper, we will study finite difference approximations [3, 4, 5] to the following Dirichlet boundary value problem for the Poisson equation:

$$\begin{cases} -\Delta u(x,y) = f(x,y), & (x,y) \in \Omega, \\ u(x,y) = g(x,y), & (x,y) \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is an open bounded domain with a smooth boundary $\partial\Omega$. In this paper, for simplicity, we will set Ω to be the unit square, i.e., $\Omega = (0,1)^2$, and $g(x,y) = 0$ on $\partial\Omega$. It is well-known that the Poisson equation is an elliptic partial differential equation which has a wide range of applications in theoretical physics and mechanical engineering.

We will review two popular finite difference schemes [4] for solving the Poisson problem (1.1). The first one is the standard five-point finite difference discretization. We will also give an error analysis to show that the accuracy of the resulting difference scheme is $O(h^2)$ in the 2- norm, where h is the grid size of the spatial discretization. We then introduce a nine-point finite difference scheme which is fourth-order accurate $O(h^4)$.

Below we introduce some preliminaries of the finite difference discretizations which will be referred later. Consider a smooth real-valued function v defined on $[0,1]$. We define the grid points $x_i = ih$, $0 \le i \le m+1$, of the interval $[0,1]$, where $h = 1/(m+1)$ is the grid size. Let v_i denote the approximation to $v(x_i)$. Assume that $v \in C^4[0,1]$. Then, by Taylor's theorem, we have

$$v(x_{i+1}) = v(x_i) + v'(x_i)h + \frac{1}{2}v''(x_i)h^2 + \frac{1}{6}v'''(x_i)h^3 + \frac{1}{24}v^{(4)}(\xi_{i1})h^4, \tag{1.2}$$

$$v(x_{i-1}) = v(x_i) - v'(x_i)h + \frac{1}{2}v''(x_i)h^2 - \frac{1}{6}v'''(x_i)h^3 + \frac{1}{24}v^{(4)}(\xi_{i2})h^4, \tag{1.3}$$

for some $\xi_{i1} \in (x_i, x_{i+1})$ and $\xi_{i2} \in (x_{i-1}, x_i)$. Summing equations (1.2) and (1.3), we obtain

$$v(x_{i+1}) + v(x_{i-1}) = 2v(x_i) + v''(x_i)h^2 + \frac{h^4}{24}\left(v^{(4)}(\xi_{i1}) + v^{(4)}(\xi_{i2})\right) \tag{1.4}$$

which implies

$$u''(x_i) = \frac{1}{h^2}\left(u(x_i+h) - 2u(x_i) + u(x_i-h)\right) - \frac{h^2}{24}\left(u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\right) \tag{1.5}$$

Now, since $v \in C^4[0,1]$ and the value $\frac{1}{2}\left(v^{(4)}(\xi_{i1})+v^{(4)}(\xi_{i2})\right)$ is between $v^{(4)}(\xi_{i1})$ and $v^{(4)}(\xi_{i2})$, the intermediate value theorem ensures that there exists ξ_i between ξ_{i1} and ξ_{i2}, hence $\xi_i \in (x_i, x_{i+1})$, such that

$$v^{(4)}(\xi_i) = \frac{1}{2}\left(v^{(4)}(\xi_{i1}) + v^{(4)}(\xi_{i2})\right).$$

Therefore, we have the so-called second-order central difference approximation to $v''(x_i)$,

$$v''(x_i) = \frac{1}{h^2}\left(v(x_{i+1}) - 2v(x_i) + v(x_{i-1})\right) - \frac{1}{12}h^2 v^{(4)}(\xi_i), \tag{1.6}$$

for some $\xi_i \in (x_{i-1}, x_{i+1})$. The second-order central difference approximation (1.6) for $v''(x_i)$ will be used very often in the next two sections.

The remainder of this paper is organized as follows. In Section 2, we introduce the five-point finite difference scheme, where an error analysis in the 2-norm is also given. In Section 3, a nine-point finite difference scheme is derived. Numerical experiments are performed in Section 4 to demonstrate the theoretical analysis. Finally, some conclusions are made in Section 5.

2. A five-point finite difference scheme with an error analysis

We consider the 2-D Poisson equation with the homogeneous Dirichlet boundary condition,

$$\begin{cases} -\Delta u(x,y) = f(x,y), & (x,y) \in \Omega := (0,1)\times(0,1), \\ \quad u(x,y) = g(x,y), & (x,y) \in \partial\Omega, \end{cases} \tag{2.1}$$

To discretize the equation, we will use the uniform Cartesian grid. Let $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$ and $0 = y_0 < y_1 < \cdots < y_m < y_{m+1} = 1$ be the uniform partitions. Then the grid size is given by $h := 1/(m+1)$ and $x_i = ih$ and $y_j = jh$.

Let u_{ij} represent an approximation to $u(x_i, y_j)$ and $f_{ij} := f(x_i, y_j)$. Applying the second-order central difference discretization (1.6) to approximate problem (2.1) at the grid point (x_i, y_j), we obtain

$$\frac{1}{h^2}\left(-u_{i-1,j} + 2u_{ij} - u_{i+1,j}\right) + \frac{1}{h^2}\left(-u_{i,j-1} + 2u_{ij} - u_{i,j+1}\right) = f_{ij}. \tag{2.2}$$

That is,

$$\frac{1}{h^2}\left(-u_{i-1,j}-u_{i+1,j}-u_{i,j-1}-u_{i,j+1}+4u_{ij}\right)=f_{ij}\,. \tag{2.3}$$

In other words, we have an $m^2\times m^2$ linear system of m^2 unknowns u_{ij} for $1\le i\le m$, $1\le j\le m$. More precisely, if we employ the rowwise ordering and let

$$\mathbf{u}=\begin{bmatrix} u^{[1]} \\ u^{[2]} \\ \vdots \\ u^{[m]} \end{bmatrix}_{m^2\times 1}, \quad u^{[j]}=\begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{mj} \end{bmatrix}_{m\times 1}, \quad \mathbf{F}=\begin{bmatrix} f^{[1]} \\ f^{[2]} \\ \vdots \\ f^{[m]} \end{bmatrix}_{m^2\times 1}, \quad f^{[j]}=\begin{bmatrix} f_{1j} \\ f_{2j} \\ \vdots \\ f_{mj} \end{bmatrix}_{m\times 1}, \tag{2.4}$$

and

$$T=\begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix}_{m\times m}, \tag{2.5}$$

where all unspecified entries are 0. Then we have the following linear system problem:

$$\mathbf{Au}=\mathbf{F}, \tag{2.6}$$

with

$$\mathbf{A}=\frac{1}{h^2}\begin{bmatrix} T & -I & & & \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T & -I \\ & & & -I & T \end{bmatrix}_{m^2\times m^2}, \tag{2.7}$$

where $\mathbf{A}$ is a sparse, symmetric and nonsingular matrix. The existence of the inverse, $\mathbf{A}^{-1}$, can be verified later. Solving the linear system (2.6), we obtain a finite difference solution of (2.1).

We now give an error analysis in the 2-norm of the five-point finite difference scheme (2.2), or equivalently, the linear system (2.6). First, the local truncation error τ_{ij} at the grid point (x_i, y_j) is defined by

$$\tau_{ij}:=\frac{1}{h^2}\left(-u(x_{i-1},y_j)-u(x_{i+1},y_j)-u(x_i,y_{j-1})-u(x_i,y_{j+1})+4u(x_i,y_j)\right)-f(x_i,y_j)\,. \tag{2.8}$$

By the Taylor expansion similar to (1.2) and (1.3), we can show that

$$\tau_{ij} := -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4}(x_i, y_j) + \frac{\partial^4 u}{\partial y^4}(x_i, y_j)\right) + O(h^4) \tag{2.9}$$

and

$$\mathbf{A}\mathbf{u}^{ex} = \mathbf{F} + \boldsymbol{\tau}, \tag{2.10}$$

where $\mathbf{u}^{ex}$ is the exact vector with the components of the exact solution $u(x, y)$ evaluated at the grid points. Letting the global error $e_{ij} := u(x_i, y_j) - u_{ij}$ and noting that $\mathbf{A}\mathbf{u} = \mathbf{F}$, we obtain the error equation

$$\mathbf{A}\mathbf{e} = \boldsymbol{\tau}, \tag{2.11}$$

where $\mathbf{e}$ is the error vector with components e_{ij}. The error equation (2.11) leads to

$$\mathbf{e} = \mathbf{A}^{-1}\boldsymbol{\tau}. \tag{2.12}$$

Thus, the five-point finite difference scheme (2.2) will be globally second order accurate in some grid function norm provided that $\|\mathbf{A}^{-1}\|$ is uniformly bounded as $h \to 0^+$ (cf. [4]).

Next, we consider the 2-norm for the discretization matrix $\mathbf{A}$. By further computations (cf. [2, 4, 5]), one can show that for $p, q = 1, 2, \cdots, m$, the eigenvector $\mathbf{u}^{p,q}$ of $\mathbf{A}$ has the m^2 elements,

$$u_{ij}^{p,q} = \sin(p\pi ih)\sin(q\pi ih)$$

and the corresponding eigenvalue is

$$\lambda_{p,q} = -\frac{2}{h^2}\{(\cos(p\pi h) - 1) + (\cos(q\pi h) - 1)\} > 0.$$

Note that $h = 1/(m+1)$. Thus, the one closest to origin is $\lambda_{1,1} = 2\pi^2 + O(h^2)$, where we use the Taylor expansion, $\cos(x) = 1 - x^2/2! + x^4/4! - \cdots$. The spectral radius of $\mathbf{A}^{-1}$ is

$$\rho(\mathbf{A}^{-1}) = \frac{1}{\lambda_{1,1}} \approx \frac{1}{2\pi^2} \quad \text{as} \quad h \to 0^+, \tag{2.13}$$

and then as $h \to 0^+$, we have from [1] that

$$\|\mathbf{A}^{-1}\|_2 = \sqrt{\rho(\mathbf{A}^{-T}\mathbf{A}^{-1})} = \sqrt{\rho((\mathbf{A}^{-1})^2)} = \sqrt{(\rho(\mathbf{A}^{-1}))^2} = \rho(\mathbf{A}^{-1}) \approx \frac{1}{2\pi^2}, \tag{2.14}$$

which is uniformly bounded in h. Now, from (2.12), we have

$$\|\mathbf{e}\|_2 = \|\mathbf{A}^{-1}\boldsymbol{\tau}\|_2 \le \|\mathbf{A}^{-1}\|_2\|\boldsymbol{\tau}\|_2 \le C\|\boldsymbol{\tau}\|_2. \tag{2.15}$$

Notice that the 2-norm of the truncation error vector $\boldsymbol{\tau}$, which is a 2-D grid function [4], can be calculated by

$$\|\boldsymbol{\tau}\|_2^2 = h^2\sum_{j=1}^{m}\sum_{i=1}^{m}\tau_{ij}^2 = h^2m^2O(h^4) = h^2\left(\frac{1-h}{h}\right)^2 O(h^4) = O(h^4) \tag{2.16}$$

which combining with (2.15) implies

$$\|e\|_2 \le C\|\boldsymbol{\tau}\|_2 = O(h^2)\,. \tag{2.17}$$

In summary, we have proved the following theorem:

Theorem 2.1. *The accuracy of the five-point finite difference scheme* (2.2) *is* $O(h^2)$.

3. A nine-point finite difference scheme

In this section, we are going to derive a nine-point finite difference scheme with a higher accuracy of $O(h^4)$ for approximating the Poisson problem (2.1) with a sufficiently smooth source function f. The basic idea is similar to that in [6]. First, from the central difference approximation (1.6), we can show that

$$\begin{aligned}&\frac{1}{h^2}\Big(-u(x_{i-1},y_j)-u(x_{i+1},y_j)-u(x_i,y_{j-1})-u(x_i,y_{j+1})+4u(x_i,y_j)\Big)\\&=-\Delta u(x_i,y_j)-\frac{1}{12}h^2\frac{\partial^4 u}{\partial x^4}(x_i,y_j)-\frac{1}{12}h^2\frac{\partial^4 u}{\partial y^4}(x_i,y_j)+O(h^4).\end{aligned} \tag{3.1}$$

Adding the term $-\frac{2}{12}h^2\frac{\partial^4 u}{\partial y^2\partial x^2}(x_i,y_j)$ to the both sides of (3.1), we have

$$\begin{aligned}&\frac{1}{h^2}\Big(-u(x_{i-1},y_j)-u(x_{i+1},y_j)-u(x_i,y_{j-1})-u(x_i,y_{j+1})+4u(x_i,y_j)\Big)-\frac{2}{12}h^2\frac{\partial^4 u}{\partial y^2\partial x^2}(x_i,y_j)\\&=-\Delta u(x_i,y_j)-\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial y^2\partial x^2}+\frac{\partial^4 u}{\partial y^4}\right)(x_i,y_j)+O(h^4)\\&=-\Delta u(x_i,y_j)-\frac{1}{12}h^2\Delta f(x_i,y_j)+O(h^4)\end{aligned} \tag{3.2}$$

where we have used the identity $-\Delta u = f$ in Ω and this implies that

$$\begin{aligned}&\frac{1}{h^2}\Big(-u(x_{i-1},y_j)-u(x_{i+1},y_j)-u(x_i,y_{j-1})-u(x_i,y_{j+1})+4u(x_i,y_j)\Big)\\&-\frac{2}{12}h^2\frac{\partial^4 u}{\partial y^2\partial x^2}(x_i,y_j)+\frac{1}{12}h^2\Delta f(x_i,y_j)=-\Delta u(x_i,y_j)+O(h^4).\end{aligned} \tag{3.3}$$

Applying the second-order finite difference approximation (1.6) to the term "$\frac{\partial^4 u}{\partial y^2\partial x^2}(x_i,y_j)$" for several times, we can obtain

$$\begin{aligned}
&\frac{1}{h^2}\left(-u(x_{i-1},y_j)-u(x_{i+1},y_j)-u(x_i,y_{j-1})-u(x_i,y_{j+1})+4u(x_i,y_j)\right)\\
&-\frac{h^2}{6h^4}\left(u(x_{i-1},y_{j-1})-2u(x_{i-1},y_j)+u(x_{i-1},y_{j+1})-2u(x_i,y_{j-1})+4u(x_i,y_j)-2u(x_i,y_{j+1})\right.\\
&\left.+u(x_{i+1},y_{j-1})-2u(x_{i+1},y_j)+u(x_{i+1},y_{j+1})\right)+O(h^4)+\frac{1}{12}h^2\Delta f(x_i,y_j)\\
&=-\Delta u(x_i,y_j)+O(h^4). \qquad (3.4)
\end{aligned}$$

Combining (3.4) with the Poisson equation (2.1) evaluated at the grid point (x_i, y_j) leads to the following compact difference scheme:

$$\begin{aligned}
&-\frac{1}{6h^2}\left(4u_{i-1,j}+4u_{i+1,j}+4u_{i,j-1}+4u_{i,j+1}+u_{i-1,j-1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i+1,j+1}-20u_{ij}\right)\\
&=f_{ij}-\frac{1}{12}h^2\Delta f(x_i,y_j), \qquad (3.5)
\end{aligned}$$

which is a nine-point finite difference scheme for the Poisson problem (2.1) with local truncation error $O(h^4)$. Finally, we remark that the term "$-\frac{1}{12}h^2\Delta f(x_i,y_j)$" on the right-hand side of (3.5) can be exactly computed or approximated by the second-order central difference approximation given in the left-hand side of (2.2).

4. Numerical experiments

In this section, we will give a numerical example to illustrate the two finite difference schemes (2.3) and (3.5) introduced in Section 2 and Section 3, respectively. Let $\Omega=(0,1)\times(0,1)$ be the unit square in $\mathbf{R}^2$. We consider the following homogeneous Poisson problem:

$$\begin{cases}-\Delta u=\pi^2\left(5\cos(2\pi x)-1\right)\sin(\pi y) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \qquad (4.1)$$

Then one can check that the exact solution to problem (4.1) is given by

$$u(x,y)=\left(\cos(2\pi x)-1\right)\sin(\pi y). \qquad (4.2)$$

We write MATLAB codes of the second-order five-point finite difference scheme (2.3) and the fourth-order nine-point finite difference scheme (3.5) with $h=2^{-i}, i=3,4,...,7$, and then estimate the orders of convergence of the schemes.

The numerical results are reported in Table 4.1 and Table 4.2, where we measure the errors in the 1-norm, 2-norm, and ∞-norm [4]. From the results reported in Table 4.1

and Table 4.2, we can find that the accuracy of the five-point finite difference scheme (2.3) is $O(h^2)$, while the accuracy of the nine-point finite difference scheme (3.5) is $O(h^4)$. Numerical results confirm the theoretical analysis. The elevation plots of the exact and numerical solutions with $h = 1/16$ and $h = 1/32$ are also depicted in Figure 4.1. From Figure 4.1, we can observe that both schemes can produce accurate numerical results, even on a rather coarse grid.

Table 4.1. Error behavior of the five-point finite difference scheme (2.3)

$1/h$	$\|\mathbf{e}\|_1$	order	$\|\mathbf{e}\|_2$	order	$\|\mathbf{e}\|_\infty$	order
8	2.0066e-02	—	2.9214e-02	—	7.0623e-02	—
16	5.0037e -03	2.00	7.1637e-03	2.03	1.7319e-02	2.03
32	1.2467e-03	2.00	1.7823e-03	2.01	4.3091e-03	2.01
64	3.1172e-04	2.00	4.4502e-04	2.00	1.0760e-03	2.00
128	7.7928e-05	2.00	1.1122e-04	2.00	2.6892e-04	2.00

Table 4.2. Error behavior of the nine-point finite difference scheme (3.5)

$1/h$	$\|\mathbf{e}\|_1$	order	$\|\mathbf{e}\|_2$	order	$\|\mathbf{e}\|_\infty$	order
8	8.6294e-04	—	1.3042e-03	—	3.2231e-03	—
16	5.4049e-05	3.99	7.8982e-05	4.05	1.9510e-04	4.05
32	3.3774e-06	4.00	4.8972e-06	4.01	1.2097e-05	4.01
64	2.1102e-07	4.00	3.0546e-07	4.00	7.5453e-07	4.00
128	1.3186e-08	4.00	1.9082e-08	4.00	4.7135e-08	4.00

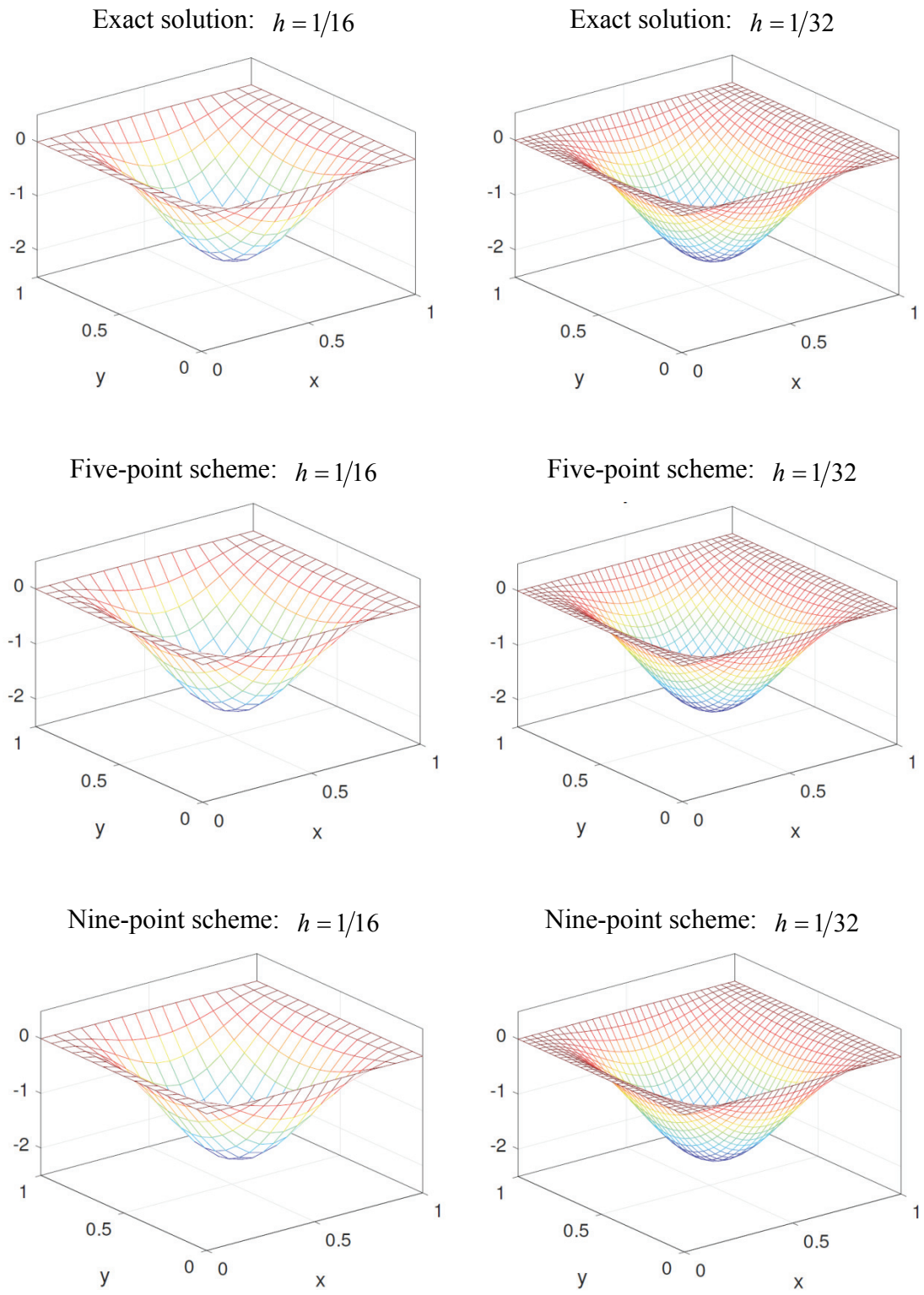

Figure 4.1: The exact and numerical solutions with $h = 1/16$ and $h = 1/32$.

5. Conclusions

In this paper, we have reviewed two popular finite difference schemes for solving the Poisson equation in 2-D domains. We have introduced the standard five-point discretization scheme. An error analysis is also given to show that the accuracy of the five-point scheme is $O(h^2)$. We have also introduced a nine-point finite difference scheme with a local truncation error $O(h^4)$, but the rigorous error analysis of this nine-point scheme is not addressed in this paper. Numerical experiments have been performed to demonstrate the orders of convergence of the schemes.

References

[1] R. A. Horn and C. R. Johnson, 1985, *Matrix Analysis*, Cambridge University Press, Cambridge.

[2] D. Kincaid and W. Cheney, 2002, *Numerical Analysis: Mathematics of Scientific Computing*, Third Edition, Brooks/Cole, Pacific Grove, CA.

[3] A. Iserles, 2009, *A First Course in the Numerical Analysis of Differential Equations*, Second Edition, Cambridge University Press, Cambridge.

[4] R. J. LeVeque, 2007, *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady State and Time Dependent Problems*, SIAM, Philadelphia.

[5] G. D. Smith, 1985, *Numerical Solution of Partial Dierential Equations: Finite Dierence Methods*, Third Edition, Oxford University Press, Oxford.

[6] W. F. Spotz, December 1995, *High-Order Compact Finite Difference Schemes for Computational Mechanics*, Ph.D. Dissertation, the University of Texas at Austin.