分數階牛頓-勞伯尼克系統混沌分析

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摘　要

近年來分數階系統動力學受到長足的關注。本文針對分數階牛頓-勞伯尼克系統的動力學作數值研究。系統顯示許多有趣的動力行為，例如固定點，週期運動、混沌運動和暫態混沌，在分數階系統中當階次小於 3 時發現混沌的存在。本研究系統能產生混沌的最低階次為 2.82，在分數階系統中同時也發現到混沌週期 2 的軌跡。

關鍵字：分數階，牛頓-勞伯尼克系統，混沌運動，暫態混沌

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CHAOS IN THE NEWTON-LEIPNIK SYSTEM WITH FRACTIONAL ORDER

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Abstract

The dynamics of fractional-order systems has attracted increasing attention in recent years. In this paper, the dynamics of the Newton–Leipnik system with fractional order was studied numerically. The system displays many interesting dynamic behaviors, such as fixed points, periodic motions, chaotic motions, and transient chaos. It was found that chaos exists in the fractional-order system with order less than 3. In this study, the lowest order for this system to yield chaos is 2.82. A period-doubling route to chaos in the fractional-order system was also found.

Key Words：fractional-order, Newton-Leipnik system, chaotic motions, transient chaos

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1. Introduction

In recent years, numerous studies and applications of fractional-order systems in many areas of science and engineering have been presented [1,2]. This is a result of better understanding of the potential of fractional calculus revealed by problems such as viscoelasticity and damping, chaos, diffusion, wave propagation, percolation and irreversibility. Recently, many systems have been identified that display fractional-order dynamics, such as viscoelastic systems [3–5], dielectric polarization [6], electrode–electrolyte polarization [7], electromagnetic waves [8], quantitative finance [9], quantum evolution of complex system [10], and the control of fractional-order dynamic systems [11–13]. Besides, Zaslavsky [14] presented a broad review of existing models of fractional kinetics and their connection to dynamical models, phase-space topology, and other characteristics of chaos. Moreover, in the last decade many researchers have found that chaotic attractors indeed exist in fractional-order systems [15–20]. In 2004, Li and Chen [21] also found the hyperchaos in fractional-order Ro·· ssler equations with order as low as 3.8. Sheu et al. [22] studied the dynamics of a new system with a fractional order. The lowest order for this new system to yield chaos was 2.43. The system displays rich dynamic behaviors, including fixed points, periodic motions, chaotic motions, and transient chaos. Moreover, it would be interesting to determine whether the system can generate a four-scroll chaotic attractor. Recently, Sheu et al. [23] investigated the influence of fractional order of damping on the dynamic behavior of the Duffing equation.

In 1981, Leipnik and Newton [24] found two strange attractors in rigid body motion in a pioneering report on the concept of chaotic motion in gyros. Very recently, this system was termed the Newton–Leipnik system by Wang and Tian [25]. Since Leipnik and Newton's work, the chaotic dynamics in rigid body motion have been intensively studied by many scientists [26–30]. Recently, in a study of the anti-control of chaos in rigid body motion, Chen and Lee [31] introduced a new chaotic system that can generate a two-scroll chaotic attractor. In 2002, Richter [32] investigated the stability and chaos control of the Newton–Leipnik system with a static non-linear feedback law based on the Lyapunov function. In 2005, Wang and Tian [25] also studied bifurcation of the Newton–Leipnik system and controlled the system using a

simple linear controller. In this paper, the dynamics of the Newton–Leipnik system with fractional order is studied numerically.

2. Fractional derivatives and the Newton－Leipnik system with fractional order

There are several definitions of fractional derivatives [1]. In this study, we use the Caputo-type fractional derivative defined by [33]

$$D_*^{\alpha} y(x) = J^{m-\alpha} y^{(m)}(x), \alpha > 0, \tag{1}$$

where $m = [\alpha]$ is the value α rounded up to the nearest integer, $y^{(m)}$ is the ordinary mth derivative of y, and:

$$J^{\beta} z(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t) dt \tag{2}$$

is the Riemann–Liouville integral operator of order $\beta > 0$, where $\Gamma(\beta)$ is the gamma function. The definition is significantly different from the classical definition of the derivative. Many other researchers use Riemann–Liouville fractional derivatives, defined by:

$$D^{\alpha} y(x) = \frac{d^m}{dx^m} J^{m-\alpha} y(x). \tag{3}$$

Typically, homogeneous initial conditions are required. The Caputo version was chosen specifically because inhomogeneous initial conditions are allowed.

The Newton–Leipnik system is described by the following non-linear differential equations:

$$\begin{aligned} \dot{x} &= -ax + y + 10yz \\ \dot{y} &= -x - 0.4y + 5xz \\ \dot{z} &= bz - 5xy \end{aligned} \tag{4}$$

where x, y, and z are the state variables, and a and b are positive parameters. It is very interesting that the Newton–Leipnik system is a chaotic system with two strange attractors [24]. When (a, b) = (0.4, 0.175), with initial states (0.349, 0, -0.16) and (0.349, 0,-0.18), system (4) displays two strange attractors. An in-depth discussion of the system can be found elsewhere [24].

Here, the fractional order of the Newton–Leipnik system is considered. The standard derivative is replaced by a fractional derivative as follows:

$$\begin{aligned} \frac{d^{q1}x}{dt^{q1}} &= -ax + y + 10yz \\ \frac{d^{q2}y}{dt^{q2}} &= -x - 0.4y + 5xz \\ \frac{d^{q3}z}{dt^{q3}} &= bz - 5xy \end{aligned} \tag{5}$$

where $0 < q_1, q_2, q_3 \leqq 1$; the order is denoted by $q = (q_1, q_2, q_3)$ here.

3. Numerical simulations

An efficient method for solving system (5) is the predictor–correctors scheme, which is a generalization of the Adams–Bashforth–Moulton scheme. In the following simulations, this method is used to integrate system (5). The detailed algorithm of the scheme was developed by Diethelm et al. [34–37]. Li and Peng [38] also used this method to study the chaos in Chen's system with fractional order. The scheme was also adopted to simulate the chaos in anew system with fractional order by Sheu et al. [22]. Applying the predictor–correctors scheme, it was found that chaos indeed exists in the Newton–Leipnik system with fractional order.

For convenience, the system parameters are given as $a = 0.4$, and $b = 0.175$, with initial state (0.349, 0, -0.18) throughout the paper. It must be pointed out when the initial state (0.349, 0, -0.16) is used, the system still eventually approaches the same attractor. Two cases are considered, as follows:

(1) Fix $q_1 = q_2 = q_3 = \alpha$, and let α vary. The system is calculated numerically against $\alpha \in [0.92, 1]$, while the incremental value of α is 0.01. The simulation results demonstrate that chaos indeed exists in the fractional-order system with order less than 3. It was found that when $0.94 \leq \alpha \leq 1.0$, system (5) shows chaotic behavior. When $\alpha =$ 0.99, 0.96 and 0.94, chaotic attractors are found; $x - y$ phase diagrams are shown in Fig. 1a–c, respectively. When $\alpha = 0.92$, the chaotic motion disappears and the system is stabilized to a fixed point, as shown by the $x - y$ phase plot in Fig. 1d. It is obvious that the trajectory for $\alpha = 0.92$ is attracted to a fixed point. A particular phenomenon of the system is that it displays transient chaotic behavior. The phase trajectory is shown in Fig. 2a for $\alpha = 0.93$. The time history of $x(t)$ is also presented in Fig. 2b, from which it is evident that the time history eventually converges to a fixed point.

(2) Fix $q_1 = q_3 = 1$, and let q_2 reduce to values less than 1. System (5) is chaotic

when $q_2 = 0.89$, and $0.96 \leq q_2 \leq 1.0$. Fig. 3a–d shows the phase trajectories for $q_2 = 0.97$, 0.89, 0.88, and 0.87, respectively. It is evident in Fig. 3a and b that system (5) displays chaotic motion. The periodic motion and a fixed point are also plotted in Fig. 4c and d, respectively. System (5) is periodic when $q_2 = 0.88$, and $0.90 \leq q_2 \leq 0.95$. Furthermore, Fig. 4a–d shows that the system can display period-1, period-2, period-4 and period-8 motions. Thus, Fig. 4 identifies a period-doubling route to chaos. In addition, it is interesting to observe that the chaotic attractors seem to rotate by approximately 90°. In other words, another strange attractor is also present in this case

From the above analysis, the lowest order we found for this system to yield chaos is 2.82. The system displays a rich dynamic behaviors, such as fixed points, periodic motions, chaotic motions, and transient chaos. Besides, two strange attractors are present due to different fractional orders.

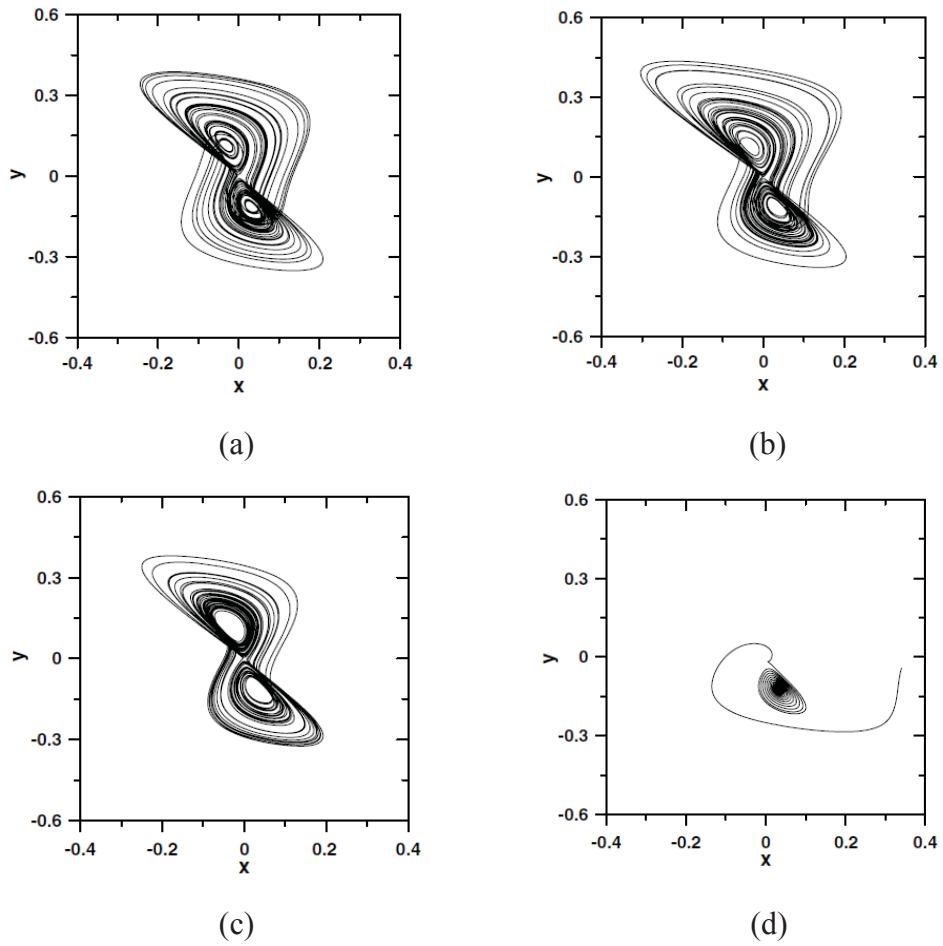
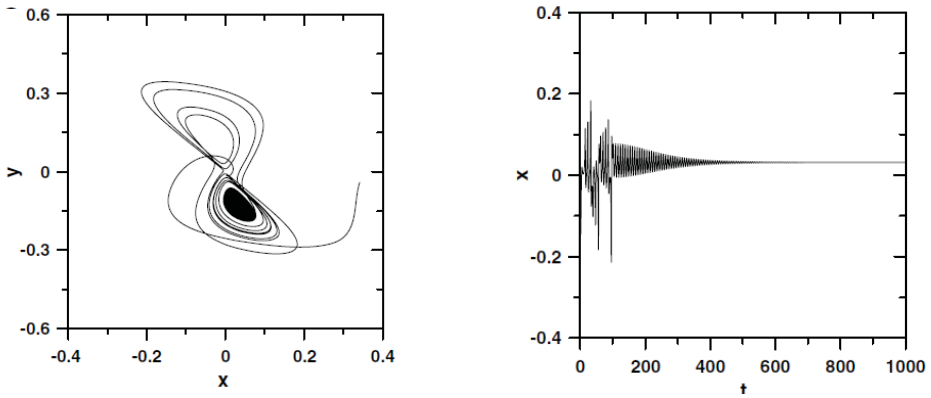


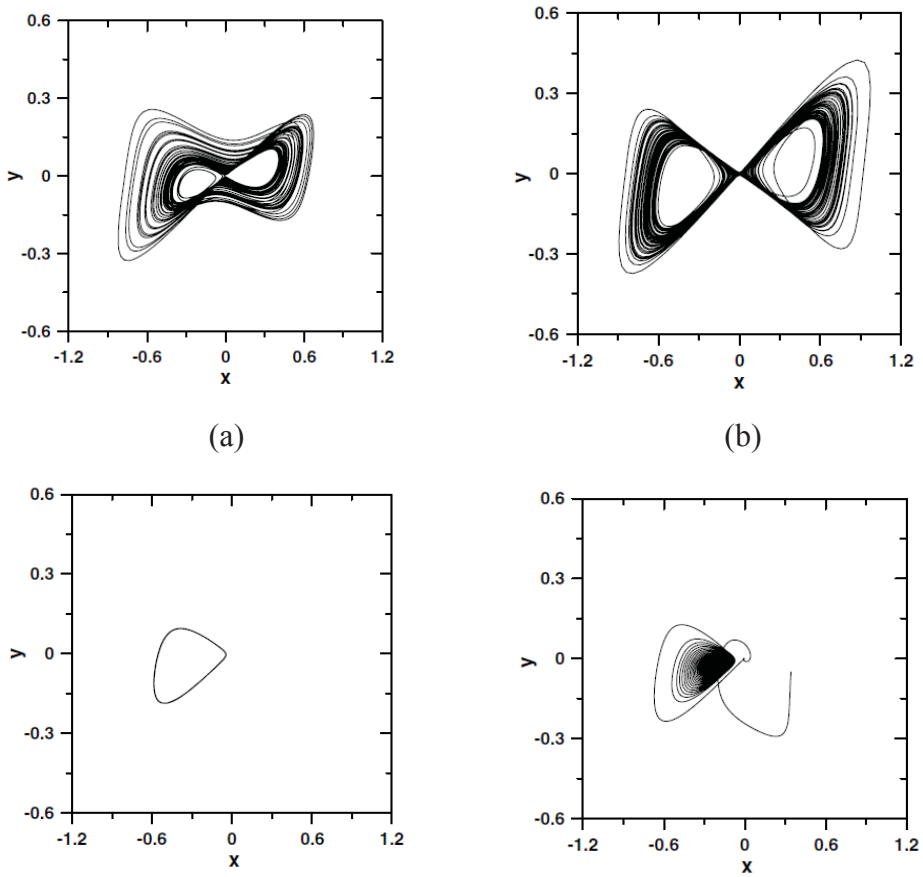
Fig.1. Phase diagrams for system with fractional orders at $q_1 = q_2 = q_3 = \alpha$: (a) $\alpha = 0.99$; (b) $\alpha = 0.96$; (c) $\alpha = 0.94$; and (d) $\alpha = 0.92$.



(a) (b)

Fig.2. Dynamic behavior of system (5) at $q_1 = q_2 = q_3 = \alpha = 0.93$:

(a) phase diagrams; and (b) time history of x(t).



(a) (b)

(c) (d)

Fig.3. Phase diagrams for system (5) at $q_1 = q_3 = 1$ and:

(a) $q_2 = 0.97$; (b) $q_2 = 0.89$; (c) $q_2 = 0.88$; and (d) $q_2 = 0.87$.

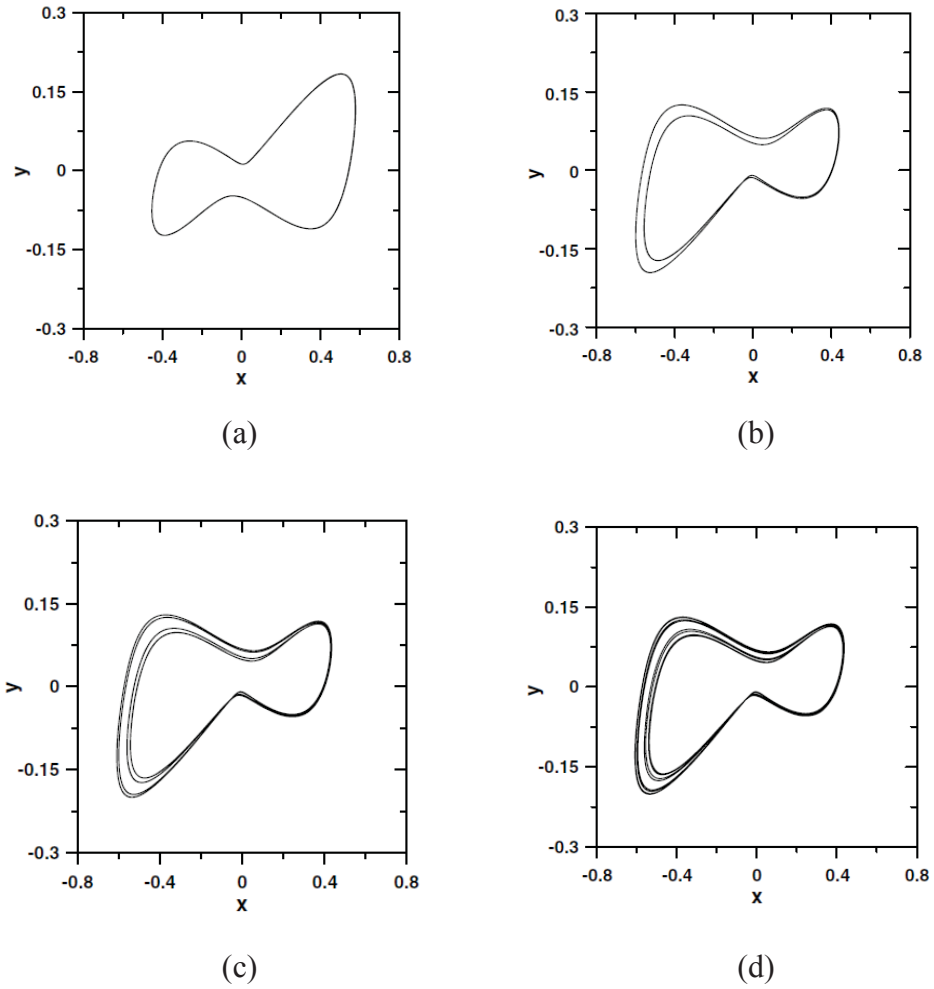


Fig.4. Phase diagrams for system (5) at $q_1 = q_3 = 1$ and:
(a) $q_2 = 0.94$; (b) $q_2 = 0.95$; (c) $q_2 = 0.953$; and (d) $q_2 = 0.9537$.

4. Conclusions

The dynamics of the Newton–Leipnik system with fractional order was studied numerically. The system was found to display rich dynamic behaviors, such as fixed points, periodic motions, chaotic motions, and transient chaos. It was found that chaos exists in the fractional-order system with order less than 3. The lowest order found to have chaos was 2.82. A period-doubling route to chaos in the fractional-order system was also found. The fact that in the second case examined, the chaotic attractors seem to rotate by approximately 90° is of particular interest. In other words, two strange attractors were also found in this study. It is known that fractional derivatives still lack complete geometrical interpretations in the time domain. Therefore, theoretical analysis

of the dynamics of the fractional-order system should be the subject of future studies. On the other hand, chaos control and synchronization of this system are still interesting problems to be investigated and should also be considered in the near future.

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