用分數微積分法解拉氏微分方程式

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摘　要

近幾十年來，分數微積分廣泛的被研究而且愈來愈重要且受歡迎。許多學者用分數微積分解決了許多物理科學方面的問題，而且分數微積分已被證明廣泛的應用在物理、工程、流體力學、黏彈性等應用科學上。由物理現象所產生的許多不同問題，衍生出各種不同的偏微分方程式。傳統方法在解決數學物理的邊界值問題是用傅立葉轉換法、還有其他的各種積分轉換法來解。本文在利用分數微積分的方法來解各種工程問題之封閉式解。即用分數微積分的方法解決一般二階偏微分方程（如熱方程式、拉氏方程式）的一些問題。

關鍵字：分數微積分、熱方程式、拉氏方程式

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SOLUTION OF LAPLACE' S EQUATION BY MEANS OF FRACTIONAL CALCULUS

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Abstract

During the past three decades or so, the widely-investigated subject of fractional calculus (that is, calculus of derivatives and integrals of any arbitrary real or complex order) has remarkably gained importance and popularity due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering. Recently, many problems in the physical sciences can be expressed and solved succinctly by recourse to the fractional calculus. Various problems which arise from the physical situation lead to certain classes of partial differential equations. The classical methods in obtaining solutions of the boundary value problems of mathematical physics are Fourier transform, and other integral transforms. The main object of this paper is by using the method of Fractional Calculus to get the closed solution of various engineering problems. That is we use the method of fractional calculus to solve the Partial Differential Equations, such as heat equation, and Laplace’s equation.

Key Words and Phrases. Fractional calculus, Heat equation, Laplace’s equation

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1. INTRODUCTION

We first introduction the definitions, lemma, and theorems of fractional calculus:

(1) Definition . (by K. Nishimoto) [12]

Let $D=\left\{\underset{-}{D},\underset{+}{D}\right\}$, $C=\left\{\underset{-}{C},\underset{+}{C}\right\}$

$\underset{-}{C}$ be a curve along the cut joining two point z and $-\infty+i\ \ \mathrm{Im}(z)$,

$\underset{+}{C}$ be a curve along the cut joining two point z and $\infty+i\ \ \mathrm{Im}(z)$,

$\underset{-}{D}$ be a domain surrounded by $\underset{-}{C}$, $\underset{+}{D}$ be a domain surrounded by $\underset{+}{C}$,

(Here D contains the points over the curve C)

Moreover , let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_\nu=(f)_\nu=c(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\xi)}{(\xi-z)^{\nu+1}}d\xi \qquad \left(\nu\notin Z^-\right) , \tag{1.1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \qquad \left(m\in Z^+\right) \tag{1.2}$$

where $-\pi\le\arg(\xi-z)\le\pi$ for $\underset{-}{C}$, $0\le\arg(\xi-z)\le 2\pi$ for $\underset{+}{C}$,

$\xi\ne z$, $z\in C$, $\nu\in R$, Γ: Gamma function ,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$) , with respect to z , of the function f , if $\left|(f)_\nu\right|<\infty$.

(2) On the fractional calculus operator N^ν [13]

Theorem A. Let fractional calculus operator (Nishimoto's operator) N^ν be

$$N^\nu=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{d\xi}{(\xi-z)^{\nu+1}}\right) \qquad \left(\nu\notin Z^-\right) \text{ , [Refer to (1.1)]} \tag{1.3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \qquad \left(m \in Z^{+}\right) \ , \tag{1.4}$$

and define the binary operation $\circ$ as

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}\left(N^{\alpha} f\right) \qquad \left(\alpha, \beta \in R\right) \ , \tag{1.5}$$

then the set

$$\left\{N^{\nu}\right\} = \left\{N^{\nu} \middle| \nu \in R\right\} \tag{1.6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $\left(N^{\nu}\right)^{-1} = N^{-\nu}$ to the fractional calculus operator N^{ν}, for the function *f* such that $f \in F = \left\{ f; 0 \neq \left| f_{\nu} \right| < \infty, \nu \in R \right\}$, where $f = f\left(z\right)$ and $z \in C$. (viz. $-\infty < \nu < \infty$) .

(For our convenience , we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α})

Theorem B. The "$F.O.G.\left\{N^{\nu}\right\}$" is an " Action product group which has continuous index ν " for the set of *F*.(F.O.G.: Fractional calculus operator group)

(3) **Lemma 1** We have [12]

(i)

$$\left(\left(z-c\right)^{\beta}\right)_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma\left(\alpha-\beta\right)}{\Gamma\left(-\beta\right)} \left(z-c\right)^{\beta-\alpha} \qquad \left(\left| \frac{\Gamma\left(\alpha-\beta\right)}{\Gamma\left(-\beta\right)} \right| < \infty\right) \ , \tag{1.7}$$

(ii)

$$\left(\log\left(z-c\right)\right)_{\alpha} = -e^{-i\pi\alpha} \Gamma\left(\alpha\right)\left(z-c\right)^{-\alpha} \qquad \left(\left|\Gamma\left(\alpha\right)\right| < \infty\right) \ , \tag{1.8}$$

(iii)

$$\left(\left(z-c\right)^{-\alpha}\right)_{-\alpha} = -e^{-i\pi\alpha} \frac{1}{\Gamma\left(\alpha\right)} \log\left(z-c\right) \qquad \left(\left|\Gamma\left(\alpha\right)\right| < \infty\right) \ , \tag{1.9}$$

(iv)

$$\left(u \cdot \upsilon\right)_{\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\alpha+1\right)}{k!\Gamma\left(\alpha+1-k\right)} u_{\alpha-k} \upsilon_{k} \qquad \left(u = u(z), \upsilon = \upsilon(z)\right) \tag{1.10}$$

where $z \neq c$ in (1.7) , (1.8) and (1.9) .

2. FORMULATION

For a function $u = u(z, t)$ of two independent variables z and t, we choose to use the convenient notation:

$$\frac{\partial^{\mu+\nu} u}{\partial z^{\mu} \partial t^{\nu}} \tag{2.1}$$

to abbreviate the partial fractional differintegral of $u = u(z, t)$ of order μ with respect to z and of order ν with respect to t $(\mu, \nu \in R)$.

$$P(z;p) := \sum_{k=0}^{p} a_k z^{p-k} \quad (a_0 \neq 0; p \in N) \tag{2.2}$$

$$Q(z;q) := \sum_{k=0}^{q} b_k z^{q-k} \quad (b_0 \neq 0; q \in N) \tag{2.3}$$

$$H(z;p,q) := \int^{z} \frac{Q(\varsigma;q)}{P(\varsigma;p)} d\varsigma \quad \left(z \in C \setminus \{z_1, \cdots, z_p\}\right) \tag{2.4}$$

Theorem 1 [9]

Let the polynomials $P(z; p)$ and $Q(z; q)$ be defined by (2.2) and (2.3), respectively. Suppose also that the function $H(z; p, q)$ is given by (2.4). Then the linear partial fractional differintegral equation

$$\begin{aligned} &P(z;p)\frac{\partial^{\mu} u}{\partial z^{\mu}} + \left[\sum_{k=1}^{p-1}\binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q}\binom{\nu}{k-1} Q_{k-1}(z;q-1)\right]\frac{\partial^{\mu-k} u}{\partial z^{\mu-k}} + C\frac{\partial^{\mu-p} u}{\partial z^{\mu-p}} \\ &= A\frac{\partial^{\mu-p+2} u}{\partial z^{\mu-p}\partial t^2} + B\frac{\partial^{\mu-p+1} u}{\partial z^{\mu-p}\partial t} \quad (\mu, \nu \in R; \quad p, q \in N) \end{aligned} \tag{2.5}$$

has solutions of the form

$$u(z,t) = \begin{cases} K_1\left(e^{-H(z;p,q-1)}\right)_{\nu-\mu+1} \cdot e^{\xi t} & (A \neq 0) \\ K_2\left(e^{-H(z;p,q-1)}\right)_{\nu-\mu+1} \cdot e^{\eta t} & (A = 0; B \neq 0) \end{cases} \tag{2.6}$$

where K_1 and K_2 are arbitrary constants, A, B and C are given constants, and ξ and η are defined by

$$\xi := \frac{-B \pm \sqrt{B^2 + 4(C-D)A}}{2A}, \ (A \neq 0) \ \text{and} \ \eta := \frac{C-D}{B}, \ (A = 0; B \neq 0) \tag{2.7}$$

with

$$D := \binom{\nu}{p} p!\, a_{0,} \tag{2.8}$$

provided that the second member of (2.6) exists in each case.

Theorem 2 [11]

The second-order linear partial differential equation:

$$\begin{aligned}&(z+\alpha)(z+\beta)\frac{\partial^2 u}{\partial z^2}+\left[\left[\rho+2(\nu+1)\right]z+\sigma+(\nu+1)(\alpha+\beta)\right]\frac{\partial u}{\partial z}+Cu(z,t)\\&=A\frac{\partial^2 u}{\partial t^2}+B\frac{\partial u}{\partial t}\quad\left(z\in C\setminus\{-\alpha,-\beta\};\ \ \rho\neq 0;\ \ \nu\in R\right)\end{aligned}\tag{2.9}$$

has solutions of the form:

$$u(z,t)=\begin{cases}K_1\left((z+\alpha)^{(\sigma-\rho\alpha)/(\alpha-\beta)}\cdot(z+\beta)^{(\sigma-\rho\beta)/(\beta-\alpha)}\right)_\nu\cdot e^{\xi t}\ (A\neq 0)\\K_2\left((z+\alpha)^{(\sigma-\rho\alpha)/(\alpha-\beta)}\cdot(z+\beta)^{(\sigma-\rho\beta)/(\beta-\alpha)}\right)_\nu\cdot e^{\eta t}\ (A=0;B\neq 0)\end{cases}$$

$$\left(z\in C\setminus\{-\alpha,-\beta\};\ \ \alpha\neq\beta;\ \ \rho\neq 0;\ \ \nu\in R\right)\tag{2.10}$$

and

$$u(z,t)=\begin{cases}K_1\left((z+\alpha)^{-\rho}\cdot e^{(\sigma-\rho\alpha)/(z+\alpha)}\right)_\nu\cdot e^{\xi t}\ (A\neq 0)\\K_2\left((z+\alpha)^{-\rho}\cdot e^{(\sigma-\rho\alpha)/(z+\alpha)}\right)_\nu\cdot e^{\eta t}\ (A=0;B\neq 0)\end{cases}$$

$$\left(z\in C\setminus\{-\alpha,-\beta\};\ \ \alpha=\beta;\ \ \rho\neq 0;\ \ \nu\in R\right)\tag{2.11}$$

where K_1 and K_2 are arbitrary constants, *A*, *B* and *C* are given constants, and ξ and η are defined by (2.7) with

$$D:=(\nu+1)(\rho+\nu)$$

provided that the second members of (2.10) and (2.11) exist in each case.

3. SOLUTIONS OF SECOND-ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

We can solve the boundary-value problems involving Laplace's equation by the method of fractional calculus.

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$$

By theorem 2, let

$$A=-1,\ B=0,\ C=0,\ P(r,2)=r^2,\ Q(r,1)=\rho r,\ \rho+2(\nu+1)=1,\ \alpha=\beta=0,$$

we have

$$D=(\nu+1)(\rho+\nu)=-(\nu+1)^2$$

$$\xi=\frac{-B\pm\sqrt{B^2+4(C-D)A}}{2A}=\mp|\nu+1|i$$

(1) When $\nu+1<0,\ \text{let}\ m=-\nu-1$

$\nu=-2,-3,\cdots \quad m=1,2,\cdots$

$$u_\nu(r,\theta)=K_\nu\left(r^{-\rho}\right)_\nu\cdot e^{i\xi\theta}\ \text{where}\ \xi=\pm\text{mi}$$

$$\left(r^{-\rho}\right)_\nu=e^{-i\pi\nu}\frac{\Gamma(\nu+\rho)}{\Gamma(\rho)}r^{-\rho-\nu}$$

$$=e^{-i\pi\nu}\frac{\Gamma(-\nu-1)}{\Gamma(-2\nu-1)}r^{\nu+1}$$

$$=(-1)^{m+1}\frac{\Gamma(m)}{\Gamma(2m+1)}r^{-m}$$

So, $u_m(r,\theta)=r^{-m}\left(C_m\cos m\theta+D_m\sin m\theta\right),\quad m=1,2,3,\cdots$

We must define $C_m=0,\ D_m=0,$ for $m=1,2,3,\cdots$ in order to guarantee that the solution u is bounded at the center of the plate (which is r =0).

(2) When $\nu+1\geq 0,\ \text{let}\ n=\nu+1$

$$u_\nu(r,\theta)=K_\nu\left(r^{-\rho}\right)_\nu\cdot e^{i\xi\theta}\ \text{where}\ \xi=\pm\text{ni}$$

As $\nu=-1,\ u_{-1}=K_{-1}\left(r^{-1}\right)_{-1}=A_0+c_1\ln r$. We must define c_1= 0 in order to guarantee that the solution u is bounded at the center of the plate (which is r =0).

For ν=0,1,2,$\cdots$, $\left(r^{-\rho}\right)_\nu=e^{-i\pi\nu}\frac{\Gamma(\nu+\rho)}{\Gamma(\rho)}r^{-\rho-\nu}$

$$=e^{-i\pi\nu}\frac{\Gamma(-\nu-1)}{\Gamma(-2\nu-1)}r^{\nu+1}$$

$$=\frac{\Gamma(2n)}{\Gamma(n+1)}r^n$$

then $u_n(r,\theta)=r^n\left(A_n\cos n\theta+B_n\sin n\theta\right),\quad n=1,2,3,\cdots$. Thus

$$u(r,\theta)=A_0+\sum_{n=1}^{\infty}r^n\left(A_n\cos n\theta+B_n\sin n\theta\right),\quad n=1,2,3,\cdots$$

Conclusion, we use the method of fractional calculus to solve the Laplace's equation.

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