# DELAY TIME COMPUTATION IN INPUT-DELAY SYSTEMS FROM A RECONSTRUCTIVE DIGITAL MODEL

Chuen-Ming Chen*,　Pin-Han Li

Department of Computer Science and Information Engineering

Taipei City University of Science and Technology

No. 2, Xueyuan Rd., Beitou, 112 Taipei, Taiwan, R. O. C.

## Abstract

The input delay problem can not be ignored in system analysis and signal processing. In this paper, a method is used to convert the analog input-delay state-space equation to its corresponding counterpart of the digital model. Basing on the derived relationship between the two model equations, the computation of delay time can be obtained. The delay time is smaller than the sampling period is discussed in this investigagtion, and it is easily extended in a larger one. It is convenient and attractive for the users to apply the model conversion in system analysis. It will also support more precise consideration on the implementation of real systems.

**Key Words**: Delay system, delay time, digital model, model conversion

*Email：cm.chen@tpcu.edu.tw　Tel：02-28927154ext.8412 Fax : 02-28960674

DELAY TIME COMPUTATION IN INPUT-DELAY SYSTEMS FROM A RECONSTRUCTIVE DIGITAL MODEL

# 由重建的數位模式計算輸入延遲系統之延遲時間量

陳春明*　李品翰

臺北城市大學資訊工程系

112臺北市北投區學園路2號

## 摘要

系統分析與信號處理的輸入延遲問題是不容忽視的，在本論文中提出一種方法，將類比輸入延遲狀態空間方程式轉換成對等的數位模式，基於這兩種類型模式方程式的相互關係，延遲時間的計算能夠被獲得。在本研究中，針對小於取樣週期的延遲時間量探討，但極容易可擴展至較大的延遲時間問題。本研究可以方便且更吸引使用者去應用此模式轉換於系統分析中，並將支援更精確思慮的真實系統實現。

**關鍵詞**：延遲系統，延遲時間，數位模式，模式轉換

*Email：cm.chen@tpcu.edu.tw　Tel：02-28927154ext.8412　Fax : 02-28960674

## 1. Introduction

In order to analyze signal or design the performance in a dynamic system, a precise model is necessary that will adequately described the system motion. For the purpose of modeling system, the available information can be obtained in two principal ways. First, there are the knowledge of physics and the scientific phenomena that have developed into equations of motion to explain the dynamic responses. Second, it is often the case which involves too many complex sophistication to give a satisfactory description of the dynamic system. In the second circumstance, the designer turns to identified data by mathematical derivation and sampled-data evaluation from discrete-time systems. Thus the precise model of the real system will cause the actual dynamics to be under control and to give good performance.

Especially, it is unavoidable that a time delay arises often in many systems due to the time in transport or physical inertia [1,2]. The time delay may be imbedded in a part of the overall systems, and it is difficult to analyze continuous-time systems with time delay because systems are infinite dimensional [2]. Therefore, an equivalent discrete-time dynamic model is necessary to be used in a sampled-data system. There are many well-developed approaches [3-6] to identify the state-space formulation and the system analysis of a discrete-time model becomes much easier. However, a real signal of the physical implementation is always in analog type, the delay factor in discrete-time model of the time-delay system must be identified and returned into its originally real systems. Moreover, a system designer should be able to achieve the delay time conveniently from its discrete-time model, so the signal process of the implemented system can be reconsidered under the influence of the delay time. Furthermore, identifying the time delay from discrete-time model, the effects of delay can be detected and the accuracy of systeml design [6] also be improved.

The key point in this paper is going to derive a conveniently digital state-space equation that is transformed from its originally analog state-space equation. The derived sampled-data model is easy to be used by simple computer programming or any well-developed algorithm. Thus the input-delay time can be evaluated precisely.

## 2. A constructive digital model and the time delay

Let us consider a designed system that is implemented in analog domain, and it is described as an input-delay state-space model [6,7]

$$\dot{x}(t) = Ax(t) + Bu(t-\varphi) \tag{1}$$

where $x(t) \in R^{n\times 1}$ in the above equation is the system state, $u(t-\varphi) \in R^{m\times 1}$ is the input signal with time delay $\varphi$, and the system parameters $A$ and $B$ are of the appropriate dimensions. The corresponding discrete-time model of Eq. (1) with a zero-order hold sampler is described as

$$\begin{aligned} x(kT+T) &= e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\mu)}Bu(t-\varphi)d\mu \\ &= Gx(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\mu)}Bu(t-\varphi)d\mu \end{aligned} \tag{2}$$

where $G=e^{AT}$ is the corresponding system matrix, $T$ is the sampling period, and it is assumed that the small time delay $\varphi$ is less than one sampling period. Input signal in the discrete-time model is piecewise constant over every sampling period. The integral term in Eq. (2) can be evaluated by splitting $kT \sim kT+T$ into $kT \sim kT+\varphi$ and $kT+\varphi \sim kT+T$, Eq. (2) gives

$$\begin{aligned} x(kT+T) &= Gx(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\mu)}Bu(t-\varphi)d\mu \\ &= Gx(kT) + \left\{\int_{kT}^{kT+\varphi} e^{A(kT+T-\mu)}Bd\mu\right\} u(kT-T) \\ &\quad + \left\{\int_{kT+\varphi}^{kT+T} e^{A(kT+T-\mu)}Bd\mu\right\} u(kT) \\ &= Gx(kT) + H_1u(kT-T) + H_0u(kT) \end{aligned} \tag{3}$$

where the concerning input signal $u(kT-T)$ is input matrix $H_1$ given as

$$\begin{aligned} H_1 &= \int_{kT}^{kT+\varphi} e^{A(kT+T-\mu)}Bd\mu \\ &= \left(e^{AT} - e^{A(T-\varphi)}\right)A^{-1}B \\ &= \left(G - e^{AT-AT(\varphi/T)}\right)A^{-1}B \\ &= (G - G^{1-\gamma})A^{-1}B \end{aligned} \tag{4}$$

and the concerning input signal $u(kT)$ is input matrix $H_0$ given as

$$H_0 = \int_{kT+\varphi}^{kT+T} e^{A(kT+T-\mu)} B d\mu$$

$$=\left(e^{A(T-\varphi)} - I_n\right)A^{-1}B$$

$$=\left(e^{AT-AT(\varphi/T)} - I_n\right)A^{-1}B$$

$$=(G \cdot G^{-\gamma} - I_n)A^{-1}B$$

$$=(G^{1-\gamma} - I_n)A^{-1}B \tag{5}$$

The fractional factor $\gamma = \varphi / T$, $I_n$ denotes an $n \times n$ identify matrix. If the system matrix $A$ is singular, the computations of $H_1$ and $H_0$ that involve inverse matrix $A^{-1}$ can be carried our by the geometric-series method[6]. As shown in Eq. (3) the discrete-time formulation with state-space equation is generally used for modeling an input-delay system, and $\left(G, H_{1,} H_0\right)$ in the state equation can be easily identified by data measurement. Once the discrete-time model $\left(G, H_{1,} H_0\right)$ is available for the discrete-time signal analysis, it is desired to find the concerning $A$, $B$, and input delay $\varphi$ in Eq. (1) for the continuous-time system. The procedure is given as follows. It is easily related from $G$=$e^{AT}$, $H_1$, and $H_0$, and the continuous-time model can give

$$A = \frac{1}{T} \cdot ln\, G \tag{6}$$

The above relationship between system matrix $A$ and $G$ has been explored by several researchers [7,8]. From Eq. (4) and Eq. (5), it gives

$$H_1 + H_0 = (G - I_n)A^{-1}B \tag{7}$$

and $B$ can be obtained by

$$B = A(G - I_n)^{-1}(H_1 + H_0) \tag{8}$$

The key work of computing time delay $\varphi$ plays an important role in this research. The following expanded series is going to be approximately described for the purpose of yielding the fractional delay $\varphi$.

$$G^{(1-\gamma)} = e^{AT(1-\gamma)}$$

$$= I_n + (1-\gamma)TA + \frac{1}{2!}[(1-\gamma)TA]^2 + \sum_{j=3}^{\infty} \frac{1}{j!} [(1-\gamma)TA]^j$$

$$\cong I_n + (1-\gamma)TA + \frac{1}{2!}[(1-\gamma)TA]^2 + \sum_{j=3}^{\infty} \frac{1}{2^{(j-1)}} [(1-\gamma)TA]^j$$

$$= I_n + \left\{ I_n + \frac{1}{2}[(1-\gamma)TA] + \sum_{j=3}^{\infty} \frac{1}{2^{(j-1)}} [(1-\gamma)TA]^{j-1} \right\} (1-\gamma)TA$$

$$= I_n + \left\{ I_n - \frac{1}{2}[(1-\gamma)TA] \right\}^{-1} (1-\gamma)TA \tag{9}$$

Considering on the convergent condition in Ed.(9), it gives

$$\frac{1}{2}[(1-\gamma)T] < {}^{1}/_{\|A\|}$$

Substituting Eq. (9) into Eq. (5) yields

$$H_0 = \left[ I_n - \frac{1}{2}(1-\gamma)TA \right]^{-1} (1-\gamma)TB \tag{10}$$

or

$$H_0 - \frac{1}{2}(1-\gamma)TAH_0 = (1-\gamma)TB \tag{11}$$

Rearranging Eq. (11) to obtain

$$H_0 = (1-\gamma)T\left[B + \frac{1}{2}AH_0\right]$$

$$= (T-\varphi)[B + \frac{1}{2}AH_0]$$

$$= (T-\varphi)\, E \tag{12}$$

where $E = B + \frac{1}{2}AH_0$. By comparing the associated elements of $H_0$ and $E$ in Eq. (12), the delay time $\varphi$ can be evaluated by

$$\varphi = T - \frac{1}{n \times m} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{h_{ij}}{e_{ij}} \tag{13}$$

$h_{ij}$ and $e_{ij}$ represent the elements of the ith-row jth-column of $H_0$ and $E$, respectively.

## 3. Numerical examples

Example 1:

Let us consider a continuous-time input-delay state-space model

$$\dot{x}(t) = Ax(t) + Bu(t-\varphi)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} u(t-0.15)$$

where $\varphi = 0.15$ is the delay time concerning the above input-delay system. A sampling period $T = 0.2$ sec is considered, one has the equivalent discrete-time model in Eq. (3)

$$x(kT+T) = Gx(kT) + H_1 u(kT-T) + H_0 u(kT)$$

$$G = e^{AT} = \begin{bmatrix} 0.9988 & 0.1976 & 0.0164 \\ -0.0164 & 0.9660 & 0.1484 \\ -0.1484 & -0.3131 & 0.5209 \end{bmatrix}$$

$$H_1 = [G - G^{1-\gamma}]A^{-1}B$$

$$= \left[G - G^{\left(1-\frac{3}{4}\right)}\right] A^{-1}B$$

$$= \left[G - G^{\left(1-\frac{3}{4}\right)}\right] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1883 & 0.3603 \\ 0.3095 & 0.5017 \\ 0.0235 & 0.2826 \end{bmatrix}$$

$$H_0 = \left[G^{(1-\gamma)} - I_3\right]A^{-1}B$$

$$= \left(G^{\left(1-\frac{3}{4}\right)} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0525 & 0.1038 \\ 0.1011 & 0.1546 \\ 0.0404 & 0.1760 \end{bmatrix}$$

where the fractional number $\gamma = {}^{\varphi}/_{T} = {}^{0.15}/_{0.2} = {}^{3}/_{4}$.

Example 2:

Let us consider a discrete-time input-delay state-space model

$$x(kT+T) = Gx(kT) + H_1 u(kT-T) + H_0 u(kT)$$

as shown in Example 1. It is desired to identify the delay time in the corresponding continuous-time model. The evaluated model represented by $A^*$, $B^*$ and the time delay $\varphi$ can be approximatedly obtained by Eqs. (6) , (8), and (13)

$$A^* = \frac{1}{T} \cdot ln\, G = \frac{1}{0.2} \cdot ln \begin{bmatrix} 0.9988 & 0.1976 & 0.0164 \\ -0.0164 & 0.9660 & 0.1484 \\ -0.1484 & -0.3131 & 0.5209 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ -1.0000 & -2.0000 & -3.0000 \end{bmatrix}$$

$$B^* = A^*(G - I_n)^{-1}(H_1 + H_0)$$

$$= \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ -1.0000 & -2.0000 & -3.0000 \end{bmatrix}$$

$$\cdot \left( \begin{bmatrix} 0.9988 & 0.1976 & 0.0164 \\ -0.0164 & 0.9660 & 0.1484 \\ -0.1484 & -0.3131 & 0.5209 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

$$\cdot \left( \begin{bmatrix} 0.1883 & 0.3603 \\ 0.3095 & 0.5017 \\ 0.0235 & 0.2826 \end{bmatrix} + \begin{bmatrix} 0.0525 & 0.1038 \\ 0.1011 & 0.1546 \\ 0.0404 & 0.1760 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1.0000 & 2.0000 \\ 2.0000 & 3.0000 \\ 1.0000 & 4.0000 \end{bmatrix}$$

$$E = B^* + \frac{1}{2} A^* H_0$$

$$= \begin{bmatrix} 1.0000 & 2.0000 \\ 2.0000 & 3.0000 \\ 1.0000 & 4.0000 \end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix} 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \\ -1.0000 & -2.0000 & -3.0000 \end{bmatrix}\begin{bmatrix} 0.0525 & 0.1038 \\ 0.1011 & 0.1546 \\ 0.0404 & 0.1760 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0505 & 2.0773 \\ 2.0202 & 3.0880 \\ 0.8120 & 3.5295 \end{bmatrix}$$

The computation of delay time $\varphi^*$ is given as

$$\varphi^* = T - \frac{1}{3 \times 2}\sum_{i=1}^{3}\sum_{j=1}^{2}\frac{h_{ij}}{e_{ij}}$$

$$= 0.2 - \frac{1}{6}\left(\frac{0.0525}{1.0505} + \frac{0.1038}{2.0773} + \frac{0.1011}{2.0202} + \frac{0.1546}{3.0880} + \frac{0.0404}{0.8120} + \frac{0.1760}{3.5295}\right)$$

$$= 0.1500544$$

$$\cong \varphi. \quad (\varphi = 0.15)$$

## 4. Conclusion

A new method has been discussed to convert analog input-delay systems to its corresponding counterpart of digital model. It is well-known that the digital state-space model can be obtained by the techniques of sampled-data identification. Therefore, the designer can compute the delay time from the digital model. And the original continuous-time model also can be reversely obtained by using these approaches. It is believed that the proposed method will be helpful for many designers in the field of analog and digital model applications in system design or model analysis.

## References

[1] T. T. Liu, X. Zhang, X. Y. Gao, Feb. 2016, "Stability Analysis for Continuous-Time and Discrete-Time Genetic Regulatory Networks with Delays." *Applied Mathematical Computation*, vol. 274, pp. 628-643.

[2] B. C. Kuo , 1977, "*Digital Control Systems*. " SRL Publishing Company, Champaign, Illinois.

[3] A. Rosenvasser, Y. N., Polyakov, K. Y. and B. P. Lampe, April 1999, "Application of Laplace Transformation for Digital Redesign of Continuous Control Systems." *IEEE Trans. on Automatic Control*, vol. 44, no.4, pp. 883-886.

[4]Y. G. Sung, W. S. Jang, and J. Y. Kim, September 2018, "Negative Input Shaped Commands for Unequal Acceleration and Braking Delays of Actuators," *Journal of Dynamics Systems, Measurement, and Control*, vol. 140, Paper No. 094501.

[5]B. Wamg, J. Cheng, A. Albarakati, and H. M. Fardon, November 2017, "A Mismatched membership Function Approach to Sampled-data Stabilization for T-S Fuzzy Systems with time-varying delayed signals," *Signal Processng*, vol. 140, pp. 161-170.

[6] J. S. H. Tsai, L. S. Shieh, and J. L. Zhang, 1993, " An Improvement of the Digital Redesign Method Based on the Block-Pulse Function Approximation," *Circuits, Systems and Signal Processing*, vol. 12, no. 1, pp. 37-49.

[7] K. Ogata, 1987, *Discrete-time Control System*, Prentice-Hall, Englewood Cliffs, New Jersey.

[8] A. Veloni and N. I. Miridakis, 2018, *Digital Control Systems –Theoretical Problems and Simulation Tools*, CRC Press, Taylor & Franics Groug, New York.